

Generating charge from diffeomorphisms

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ABSTRACT: We unravel some subtleties involving the definition of sphere angular momentum charges in $\text{AdS}_q \times S^p$ spacetimes, or equivalently, R-symmetry charges in the dual boundary CFT. In the AdS_3 context, it is known that charges can be generated by coordinate transformations, even though the underlying theory is diffeomorphism invariant. This is the bulk version of spectral flow in the boundary CFT. We trace this behavior back to special properties of the p-form field strength supporting the solution, and derive the explicit formulas for angular momentum charges. This analysis also reveals the higher dimensional origin of three dimensional Chern-Simons terms and of chiral anomalies in the boundary theory.

KEYWORDS: Gauge Symmetry, AdS-CFT Correspondence, Space-Time Symmetries.

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1. Introduction

AdS_q × S^p spacetimes in string/M-theory arise as solutions of gravity coupled to a p-form field strength, as described by the Euclidean signature action¹

$$S = \frac{1}{16\pi G_{q+p}} \int d^{q+p}x \left(\sqrt{g}R + \frac{1}{2} \star G_p \wedge G_p \right) . \quad (1.1)$$

This theory admits Freund-Rubin type solutions:

$$\begin{aligned} ds^2 &= ds^2_{AdS_q} + ds^2_{S^p} \\ G_p &= Q \epsilon_{S^p} . \end{aligned} \quad (1.2)$$

¹In odd dimensions a Chern-Simons term will also play an important role (see below).

In this paper we are primarily interested in the case of asymptotically, locally, $\text{AdS}_3 \times S^p$ spacetimes. Such geometries, with suitable boundary conditions, have a local $\text{SO}(p+1)$ group of symmetries associated with isometries of the p -sphere, and corresponding conserved charges. Here we aim to give an explicit expression for these conserved charges.

To appreciate that this problem is more subtle than one might guess, observe the following. Start with the solution (1.2), which has vanishing $\text{SO}(p+1)$ charges. Now perform a simple coordinate transformation that mixes up the sphere and AdS coordinates. Since we are working in the context of a diffeomorphism invariant theory, it seems natural to expect that the $\text{SO}(p+1)$ charges will continue to vanish after the coordinate transformation. But this expectation clashes with the charges usually assigned to standard solutions of this form,² and with basic aspects of the $\text{AdS}_3/\text{CFT}_2$ duality. For example, rotating BPS black hole solutions in the D1-D5 system look locally like a coordinate transformation of $\text{AdS}_3 \times S^3$, yet carry nonzero charge [1]. In the boundary CFT description there is the phenomenon of “spectral flow”, which is a relabelling of states and symmetry generators that shifts the R-charges. The gravitational description of spectral flow is known to be a coordinate transformation of the sort we have just described [2, 3].

The resolution of this puzzle, and the route to obtaining acceptable formulas for conserved charges, involves several ingredients. The solutions described by (1.1)-(1.2) will in fact carry zero charge after a coordinate transformation — extra structure is required to induce the charges. Our two basic examples are $\text{AdS}_3 \times S^3$ and $\text{AdS}_3 \times S^2$. In the S^3 case it is crucial that G_3 also have flux on AdS_3 , while for S^2 we need to include in the action a 5-dimensional Chern-Simons term, $\int C_1 \wedge G_2 \wedge G_2$. In both cases the crux of our analysis is then a careful treatment of the p -form field strength. To obtain a satisfactory gauge invariant theory on AdS_3 after reduction on the sphere, we are forced to have C_{p-1} and/or G_p transform in a nontrivial way under $\text{SO}(p+1)$. In the S^2 case G_2 will be $\text{SO}(3)$ invariant, but C_1 will not be, and the presence of the 5-dimensional Chern-Simons term then induces the nonzero charge. In the S^3 case the charge will arise from the $\text{SO}(4)$ noninvariance of G_3 . Our treatment of these two cases will admit a generalization to $\text{AdS}_{2n-1} \times S^{2n-1}$ and $\text{AdS}_{4n-1} \times S^{2n}$.

The basic tools for obtaining gauge invariant actions have been developed in the context of M5-brane anomaly cancellation [4, 5] and consistent Kaluza-Klein sphere reductions (e.g. [6–8]), and we will adapt them for our purposes. The Kaluza-Klein procedure produces gauged supergravity theories, which contain three dimensional Chern-Simons terms for the $\text{SO}(p+1)$ gauge fields. As we discuss in the next section, the construction of charges with the desired properties follows quite straightforwardly from this three-dimensional perspective. In particular, the spectral flow behavior is linked to the fact that Chern-Simons terms are only gauge invariant up to boundary terms. The basic challenge for us will be to obtain these results directly from the higher dimensional setup without performing a Kaluza-Klein reduction.

²These charge assignments are typically made by comparing with the angular momenta of asymptotically flat solutions with a given near horizon geometry. Since angular momenta are quantized they are expected to be unchanged upon taking the near horizon limit. In this work we make no reference to auxiliary asymptotically flat solutions.

To do so we can use previous insights [5] on the higher dimensional origin of Chern-Simons terms. We should emphasize that although we will be using some of the methods developed in the context of consistent Kaluza-Klein sphere reductions, we expect our results to be valid more generally. In particular, since we will only need to make reference to the asymptotic behavior of the fields, we can allow for the presence of additional fields beyond those appearing in a consistent truncation ansatz, as long as they fall off sufficiently rapidly at infinity. This is fortunate, since in some of the cases we discuss no complete consistent truncation has so far been derived in the literature. On the other hand, it would be very worthwhile to consider explicitly the general problem with other fields included, to make precise the notion of what it means for these fields to “fall off sufficiently rapidly.” In particular, it is conceivable that such a rapid falloff is inconsistent with the equations of motion of certain fields. We view this as unlikely, insofar as our analysis yields the expected results for the charges, but it would be nice to establish this directly.

To forestall a possible confusion, we remark that there is another unrelated context in which charges can be induced by coordinate transformations. Gravity in odd-dimensional AdS spacetimes has a conformal anomaly arising from the need to regulate and subtract large volume divergences in the action [9]. The result is that the gravitational action is not invariant under all coordinate transformations, specifically those that act as a Weyl transformation of the conformal boundary metric. It is therefore not too surprising that coordinate transformations can shift the charges associated with AdS energy and angular momentum, in agreement with the expected anomalous transformation law of the stress tensor [10]. Our case is different in that our coordinate transformations will not act as Weyl transformation on the boundary, and so potential violations of diffeomorphism invariance will play no role.

2. Currents and charges in AdS₃ gravity

We begin with a discussion of currents and charges in the effective three dimensional description of an underlying higher dimensional theory.³ We work in the framework of holographic renormalization [9, 10] (for a review see [13], and for additional work on defining conserved charges see [14, 15]). Our goal will then be to reproduce these results from the higher dimensional perspective.

The relevant terms in the action for the metric and 1-form potential are

$$S = \frac{1}{16\pi G_3} \int d^3x \sqrt{g} \left(R - \frac{2}{\ell^2} \right) - \frac{ik}{4\pi} \int d^3x \text{Tr}(AdA + \frac{2}{3}A^3) + \dots + S_{\text{bdy}} . \quad (2.1)$$

We are working in Euclidean signature. The ... terms refer to contributions from other matter fields and possible higher derivative terms that will not contribute to our discussion of charges, since these are controlled by the leading long distance part of the Lagrangian. The need for various boundary terms is also indicated, and will be discussed in more detail below.

³This is based on the more complete analysis in [11]. Note that also that an analogous treatment of *nonchiral* currents appears in [12]. Here we are concerned with chiral currents, since these appear in the relevant AdS/CFT examples.

The Chern-Simons term is defined with respect to an $SU(2)$ gauge group, which either can be thought of as the isometry group of an S^2 , or as one factor in the $SO(4) \approx SU(2) \times SU(2)$ isometry group of an S^3 . Invariance of the action under large gauge transformations requires that k be an integer, which we will take to be positive. The gauge field components are given by $A = A^a \frac{i}{2} \sigma^a$.

The metric is taken to be asymptotically AdS_3 in the sense that it takes the Fefferman-Graham form

$$ds^2 = d\eta^2 + e^{2\eta/\ell} g_{\alpha\beta}^{(0)} dx^\alpha dx^\beta + g_{\alpha\beta}^{(2)} dx^\alpha dx^\beta + \dots \quad (2.2)$$

The gauge fields admit the expansion

$$A = A^{(0)} + e^{-2\eta/\ell} A^{(2)} + \dots, \quad (2.3)$$

and we choose the gauge $A_\eta = 0$.⁴

Analysis of the field equations (including the effect of Maxwell type terms) shows that $A^{(0)}$ is a flat connection; that is, the field strength corresponding to (2.3) falls off as $e^{-2\eta/\ell}$. This falloff of the field strength implies that Maxwell and higher derivative terms in the action will give no contributions to the on-shell variation of the action, since the relevant surface integrals vanish. So the analysis that follows holds in complete generality.

We define a stress tensor and current by evaluating the on-shell variation of the action. When the equations of motion are satisfied, the variation takes the form

$$\delta S = \int_{\partial AdS} d^2x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta}^{(0)} + \frac{i}{2\pi} J^{\alpha a} \delta A_\alpha^{(0)a} \right). \quad (2.4)$$

Indices are raised and lowered with the conformal boundary metric $g_{\alpha\beta}^{(0)}$. To put the variation in the above form we need to add appropriate boundary terms to the action, as was indicated in (2.1). As is well known [9, 10], the gravitational boundary term is

$$S_{\text{bndy}}^{\text{grav}} = \frac{1}{8\pi G_3} \int_{\partial AdS} d^2x \sqrt{g} \left(\text{Tr} K - \frac{1}{\ell} \right), \quad (2.5)$$

where K is the extrinsic curvature of the boundary.

For reasons that we will explain momentarily, it is also natural to include the boundary term

$$S_{\text{bndy}}^{\text{gauge}} = -\frac{k}{16\pi} \int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta} A_\alpha^a A_\beta^a. \quad (2.6)$$

With these boundary terms, the on-shell variation of the action yields

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{8\pi G\ell} \left(g_{\alpha\beta}^{(2)} - \text{Tr}(g^{(2)}) g_{\alpha\beta}^{(0)} \right) + \frac{k}{8\pi} \left(A_\alpha^{(0)a} A_\beta^{(0)a} - \frac{1}{2} A^{(0)a\gamma} A_\gamma^{(0)a} g_{\alpha\beta}^{(0)} \right) \\ J_\alpha^a &= \frac{ik}{4} \left(A_\alpha^{(0)a} - i\epsilon_\alpha^\beta A_\beta^{(0)a} \right). \end{aligned} \quad (2.7)$$

⁴Choosing a gauge is not quite as innocuous as it sounds, since this theory is anomalous under gauge transformation that are nonzero at the boundary. It is perhaps better to say that we are deciding to look just at solutions of this form.

To appreciate the role of the boundary term (2.6), work in conformal gauge: $g_{\alpha\beta}^{(0)} dx^\alpha dx^\beta = dw d\bar{w}$. Then the current is

$$J_w^a = \frac{ik}{2} A_w^{(0)a}, \quad J_{\bar{w}}^a = 0. \quad (2.8)$$

The coefficient in front of (2.6) was chosen to set to zero the anti-holomorphic component of the current. This is desirable from several points of view. First, it is a standard fact from the quantization of Chern-Simons theory (e.g. [16]) that A_w and $A_{\bar{w}}$ are canonically conjugate in the sense that the variation of the action takes the form $\delta S \sim p \delta q \sim A_w \delta A_{\bar{w}}$, which is consistent with (2.8). This means that we can adopt a variational principle in which the boundary conditions are set by $A_{\bar{w}}$ only. Fixing boundary conditions for both components of A is problematic in that there will typically not be any smooth solution consistent with the chosen boundary conditions. Second, in the context of CFTs dual to the bulk AdS₃ theory, the level k SU(2) current algebra is indeed holomorphic. In this regard, we also note that (2.8) gives the correct chiral anomaly

$$D_{\bar{w}} J_w^a = \frac{ik}{2} \partial_w A_{\bar{w}}^{(0)a}, \quad (2.9)$$

where we used $F^{(0)} = 0$.

Given the current we can define a charge. In conformal gauge the charge is simply

$$J_0^a = \oint \frac{dw}{2\pi i} J_w^a = \frac{ik}{2} \oint \frac{dw}{2\pi i} A_w^{(0)a}, \quad (2.10)$$

where the contour goes around the AdS₃ boundary cylinder. The charge is therefore equivalent to the gauge holonomy. From (2.9) we see that the charge is conserved if $A_{\bar{w}}^{(0)a} = 0$. The charges obey the SU(2) Lie algebra⁵

$$[J_0^a, J_0^b] = i\epsilon^{abc} J_0^c. \quad (2.11)$$

More generally the modes $J_n^a = \oint \frac{dw}{2\pi i} w^n J_w^a$ obey an SU(2) current algebra at level k :

$$[J_n^a, J_m^b] = \frac{1}{2} m k \delta_{m,-n} \delta^{ab} + i\epsilon^{abc} J_{n+m}^c. \quad (2.12)$$

(2.7) shows that the gauge field contributes to the stress tensor as

$$T_{ww} = \frac{k}{8\pi} A_w^a A_w^a, \quad T_{\bar{w}\bar{w}} = \frac{k}{8\pi} A_{\bar{w}}^a A_{\bar{w}}^a, \quad T_{w\bar{w}} = 0, \quad (2.13)$$

in addition to the usual gravitational part. In terms of the stress tensor we define the Virasoro generator: $L_0 - \frac{c}{24} = \oint dw T_{ww}$.

Consider the shift $A_w^{(0)3} \rightarrow A_w^{(0)3} + 2\eta$. Taking w to have 2π periodicity, this induces the shift

$$\begin{aligned} L_0 &\rightarrow L_0 + 2\eta J_0^3 + k\eta^2 \\ J_0^3 &\rightarrow J_0^3 + k\eta. \end{aligned} \quad (2.14)$$

⁵The factor of $i/2\pi$ in (2.4) was chosen to bring the algebra to the standard form.

This is a so-called “spectral flow” transformation. This is an automorphism of the usual Virasoro/current algebra, and provides yet another justification for the boundary term (2.6).

We remarked earlier that in the case of an $SO(4) \approx SU(2) \times SU(2)$ gauge group we only considered one of the $SU(2)$ factors. The other factor is included as follows. We add to the action (2.1) a second Chern-Simons term with opposite sign coefficient. The boundary term analogous to (2.6) then implies that the current is purely anti-holomorphic. The explicit formulas are then essentially identical to the above, with the replacement $w \leftrightarrow \bar{w}$.

3. Higher dimensional perspective: generalities

We now turn to the higher dimensional analysis of conserved charges, and discuss some aspects of the problem common to the various cases. Some previous, but not directly related, work on a higher dimensional approach to holographic renormalization is [17–19].

We first need to discuss the class of spacetimes we will be considering. Since charges in gauge theories are expressed as surface integrals, what matters to us is the asymptotic behavior of the metric and matter fields. Our first assumption is that the metric is asymptotically, locally, $AdS_q \times S^p$, by which we mean

$$ds^2 \rightarrow ds_{AdS_q}^2 + \ell_p^2 (dy^i - A^{ij}(x)y^j)(dy^i - A^{ik}(x)y^k). \quad (3.1)$$

AdS coordinates are denoted by x . Sphere coordinates are denoted as y^i ; $i = 1 \dots p + 1$; $\sum y^i y^i = 1$. We should in principle specify the rate of falloff of fluctuations around this form, but this will not be necessary.

$SO(p + 1)$ acts on y^i in the obvious way. We identify $A^{ij}(x) = -A^{ji}(x)$ as Kaluza-Klein $SO(p + 1)$ gauge fields by noting that under an x -dependent $SO(p + 1)$ rotation of y^i , invariance of the line element (3.1) is achieved by accompanying this with the usual $SO(p + 1)$ gauge transformation of $A^{ij}(x)$.

We also need to specify the asymptotic form of the field strength G_p ; this is a good deal more subtle and is the main topic of the remainder of the paper.

Conserved charges arise as integrals of conserved currents, which are in turn defined to be conjugate to the gauge potentials A^{ij} . Specifically, the on-shell variation of the action with respect to A^{ij} takes the form of a boundary integral, which we can write as

$$\delta S = \frac{i}{4\pi} \int_{\partial AdS} d^{q-1}x \sqrt{g} J^{ij\alpha} \delta A_{\alpha}^{ij}, \quad (3.2)$$

where the boundary metric is $g_{\alpha\beta}^{(0)}$ as in (2.2), but we suppress the (0) superscript. Invariance of the action under $SO(p + 1)$ gauge transformations of A^{ij} implies covariant conservation of the current. We can then define charges by integrating the time component of the current over a spacelike hypersurface in the usual fashion. However, we should emphasize that in general the action need not be gauge invariant — variation by boundary terms is allowed — which leads to anomalous conservation laws. We will see this explicitly in the examples below.

When we specialize to the main case of interest, $AdS_3 \times S^p$ (with $p = 2, 3$), we make the further assumption that the $A^{ij}(x)$ appearing in (3.1) are flat connections; that is, the

associated field strength vanishes. This is justified as follows. For reasons that will become clear as we proceed, A^{ij} asymptotically obeys a Chern-Simons equation of motion, and this imposes flatness. Our analysis in the $AdS_3 \times S^p$ case will then proceed in parallel to that of the previous section. After adding a boundary term analogous to (2.6), the currents will take forms similar to (2.7)-(2.8), and we can define charges as before.

The main subtlety in arriving at the correct variation (3.2) lies in determining how the G_p dependent part of the action varies. We now describe the tools used in this analysis.

4. Review of global angular forms

In this section we introduce the global angular form and establish conventions and notation for dealing with sphere bundles. Our discussion follows [4, 5] but with different normalization conventions.

We will be concerned here with $AdS_q \times S^p$ as an S^p bundle over the base space AdS_q together with a connection one-form A taking values in the Lie Algebra $so(p+1)$. Sphere coordinates are denoted as y^i ; $i = 1 \cdots p+1$; $\sum y^i y^i = 1$. $SO(p+1)$ acts in the obvious way.

The connection A allows us to define vertical forms along the S^p and a curvature for the connection:

$$\begin{aligned} Dy^i &= dy^i - A^{ij} y^j \\ F^{ij} &= [D, D]^{ij} = dA^{ij} - A^{ik} \wedge A^{kj} . \end{aligned} \tag{4.1}$$

The connection A is most easily understood as representing the off diagonal components of the metric, as in (3.1). We use x to denote coordinates on the AdS base. Then $A^{ij} = A^{ij}(x)$ is a function of the AdS coordinates only.

We are most interested in the $SO(p+1)$ transformations that are implemented by a combination of a gauge transformation and a coordinate transformation. Explicitly, given an antisymmetric matrix in $so(p+1)$, $\Lambda^{ij}(x)$, we perform:

$$\begin{aligned} y^i &\rightarrow y^i + \Lambda^{ij} y^j , \\ A^{ij} &\rightarrow A^{ij} + d\Lambda^{ij} + [\Lambda, A]^{ij} . \end{aligned} \tag{4.2}$$

Dy^i and F^{ij} of course transform covariantly under (4.2).

Over any oriented S^p -bundle it is possible to uniquely define a global angular p -form, e_p , such that:

- The integral of e_p over any fiber is given by $\int_{S^p} e_p = 1$.
- $de_{2n} = 0$.
- $de_{2n-1} = \chi_{2n}$, where χ_{2n} is the Euler class of the sphere bundle.
- e_p is invariant under (4.2).

These properties make e_p well suited for writing an *ansatz* for the p -form field strength G_p supporting an $AdS_q \times S^p$ solution of supergravity [5, 7, 8].

Our main examples will concern the cases $p = 2, 3$, for which

$$\begin{aligned}
 e_2 &= \frac{1}{8\pi} \epsilon_{ijk} (Dy^i Dy^j - F^{ij}) y^k \\
 de_2 &= 0 \\
 e_3 &= \frac{1}{(2\pi)^2} \epsilon_{ijkl} \left(\frac{1}{3} Dy^i Dy^j Dy^k - \frac{1}{2} F^{ij} Dy^k \right) y^l \\
 de_3 &= \chi_4 = \frac{1}{32\pi^2} \epsilon_{ijkl} F^{ij} F^{kl} .
 \end{aligned}
 \tag{4.3}$$

It will also be useful to have an explicit expression for χ_3 , defined by $d\chi_3 = \chi_4$. This is most naturally expressed in $SU(2)_L \times SU(2)_R$ notation, as defined in the appendix. We then have, up to a closed form,

$$\chi_3 = -\frac{1}{8\pi^2} \text{Tr}(A_L dA_L + \frac{2}{3} A_L^3) + \frac{1}{8\pi^2} \text{Tr}(A_R dA_R + \frac{2}{3} A_R^3) .
 \tag{4.4}$$

4.1 Bott and Cattaneo formula

We now state a formula due to Bott and Cattaneo [20] that will prove very useful in the case of even dimensional spheres, $p = 2n$. We may write, at the level of forms:

$$\int_{S^{2n}} e_{2n} \wedge e_{2n} \wedge e_{2n} = \frac{1}{4} p_n
 \tag{4.5}$$

where p_n is the Pontrjagin class of the sphere bundle.

We now apply ‘‘anomaly descent’’ to both sides of this formula. Given an invariant closed form like e_{2n} , locally we can write $e_{2n} = de_{2n-1}^{(0)}$. The invariance of e_{2n} under (4.2) implies that $\delta e_{2n-1}^{(0)} = de_{2n-2}^{(1)}$. We proceed in analogous fashion for $p_n^{(0)}$. Then we can write

$$\int_{S^{2n}} e_{2n-1}^{(0)} \wedge e_{2n} \wedge e_{2n} = \frac{1}{4} p_n^{(0)}
 \tag{4.6}$$

up to a closed form. Note that we are not relabeling the n subscript on p by convention; $p_n^{(0)}$ is a $4n - 1$ form.

In the $n = 1$ case it is convenient to work in $SU(2)$ language by writing

$$A^a = \frac{1}{2} \epsilon^{abc} A^{bc}, \quad A = A^a \frac{i}{2} \sigma^a .
 \tag{4.7}$$

The $n = 1$ version of (4.6) is then

$$\int e_1^{(0)} \wedge e_2 \wedge e_2 = -\frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \int \text{Tr}(AdA + \frac{2}{3} A^3) .
 \tag{4.8}$$

If we now equate the gauge variations of both sides of this equation we get:

$$\int e_0^{(1)} \wedge e_2 \wedge e_2 = -\frac{1}{2} \left(\frac{1}{2\pi} \right)^2 \int \text{Tr}(\Lambda dA) .
 \tag{4.9}$$

The $n = 2$ Bott-Cattaneo formula was used in [5] to derive the Chern-Simons terms for $AdS_7 \times S^4$ spacetimes. By a similar procedure, we will use the formula for deriving the conserved charges associated with $AdS_3 \times S^2$ spacetimes, or more generally for $AdS_{4n-1} \times S^{2n}$.

5. Example: $\text{AdS}_3 \times S^2$

In this section we show how to derive the $\text{SO}(3)$ charges associated with asymptotically, locally, $\text{AdS}_3 \times S^2$ geometries. These geometries are important in string theory since they describe the near horizon limit of four dimensional black holes (e.g. [21]). We will also discuss the generalization to $\text{AdS}_{4n-1} \times S^{2n}$.

5.1 $\text{AdS}^3 \times S^2$

We begin with the action

$$S = \frac{1}{16\pi G_5} \int d^5x \left(\sqrt{g}R + \frac{1}{2} \star G_2 \wedge G_2 + i\alpha C_1 \wedge G_2 \wedge G_2 \right) . \quad (5.1)$$

As was noted earlier, the Chern-Simons term is crucial for obtaining nonzero conserved charges induced by diffeomorphisms on the sphere. We leave its coefficient α unspecified, although in specific constructions it is fixed by supersymmetry.

The metric takes the asymptotic form (3.1) with $q = 3$ and $p = 2$, and we now discuss the asymptotic form of the field strength G_2 . For $A^{ij} = 0$ we have the “background” solution with

$$G_2 = Q\epsilon_{S^2} , \quad (5.2)$$

where ϵ_{S^2} denotes the volume form on the unit 2-sphere. The question is how to modify this in the presence of nonzero A^{ij} . Here we can follow [4, 5]. We want the action for A^{ij} to be invariant, up to boundary terms, under $\delta_\Lambda A = d\Lambda + [\Lambda, A]$, in order to define a conserved current (or rather, a current that is anomalously conserved in the presence of a nonzero boundary variation). Now, since our action is diffeomorphism invariant, this invariance will be achieved provided that the action is invariant under (4.2).⁶ This in turn suggests that we should demand that G_2 be gauge invariant. Furthermore, in order to construct solutions of a fixed charge, we demand that $\int_{S^2} G_2 = 4\pi Q$, in accordance with (5.2). Finally, we must of course have $dG_2 = 0$.

These conditions lead us uniquely to:

$$G_2 = 4\pi Q e_2 , \quad (5.3)$$

with e_2 defined in (4.3). We emphasize that we are only demanding that G_2 take this form asymptotically; deep in the interior G_2 will generally deviate from this.

Although G_2 is gauge invariant, C_1 is not. Indeed, we have $C_1 = 4\pi Q e_1^{(0)}$ and

$$\delta_\Lambda C_1 = 4\pi Q d e_0^{(1)} , \quad (5.4)$$

and so the action varies by a boundary term

$$\delta_\Lambda S = -\frac{i\alpha Q^3}{2G_5} \int_{\partial \text{AdS}} \text{Tr}(\Lambda dA) , \quad (5.5)$$

⁶We will now refer to (4.2) as a gauge transformation.

where we used (4.9). We can use this variation to fix the coefficient of the Chern-Simons term in the effective three dimensional action:

$$S_{CS} = -\frac{ik}{4\pi} \int_{AdS} d^3x \text{Tr}(AdA + \frac{2}{3}A^3) . \quad (5.6)$$

with

$$k = \frac{2\pi\alpha Q^3}{G_5} . \quad (5.7)$$

In the three dimensional analysis of section 2, the currents were obtained from the Chern-Simons term. To identify the current and charges in the 5-dimensional setup we can now simply follow the analysis in section 2.

Alternatively, we can obtain the current directly by examining the on-shell variation of the action under an arbitrary variation of A . Given the form of our ansatz for G_2 , the only term in the action which contributes is the Chern-Simons term. Since the Einstein-Hilbert and G_2 kinetic terms are gauge invariant, their variations are proportional to field strengths, and we have already noted that these vanish at the boundary. The variation of the Chern-Simons term can be evaluated using the formula of Bott and Cattaneo:

$$\begin{aligned} \delta S &= \delta \left(\frac{i\alpha}{16\pi G_5} \int C_1 \wedge G_2 \wedge G_2 \right) = \frac{4i\alpha\pi^2 Q^3}{G_5} \delta \int e_1^{(0)} \wedge e_2 \wedge e_2 \\ &= \frac{ik}{4\pi} \int_{\partial AdS} \text{Tr}(A\delta A) . \end{aligned} \quad (5.8)$$

Note that this is the same formula as obtained by varying (5.6).

We can now proceed precisely as in section 2. After adding the boundary term (2.6) and going to conformal gauge, we obtain the current (2.8), and the $SU(2)$ charges (2.10). We have therefore succeeded in finding formulas for the $SO(3)$ charges with the desired properties. In particular, since flat potentials A^{ij} can yield nonzero charges, we see how charges can be induced by coordinate transformations. An explicit example of this will be given below in the $AdS_3 \times S^3$ context.

5.2 Generalization to $AdS_{4n-1} \times S^{2n}$

The preceding analysis admits a straightforward generalization. We take the action

$$S = \frac{1}{16\pi G_{6n-1}} \int d^{6n-1}x (\sqrt{g}R + \frac{1}{2} \star G_{2n} \wedge G_{2n} + i\alpha C_{2n-1} \wedge G_{2n} \wedge G_{2n}) . \quad (5.9)$$

Proceeding as above, we are led to

$$G_{2n} = \Omega_{2n} e_{2n} . \quad (5.10)$$

Application of the Bott-Cattaneo formula leads to the Chern-Simons term

$$S_{CS} = \frac{i\alpha(\Omega_{2n})^3}{64\pi G_{6n-1}} \int_{AdS} d^{4n-1}x p_n^{(0)} . \quad (5.11)$$

Varying this with respect to A^{ij} yields the currents and charges. We resist writing the resulting formulas as they are not particularly illuminating.

5.3 Comments

In the preceding it is manifest that the existence of Chern-Simons terms in the AdS theory is directly tied to the presence of such terms in the higher dimensional theory. Note that we only considered the two-derivative Chern-Simons terms in the higher dimensional theory, but in string/M-theory there can be additional Chern-Simons terms with more derivatives. These are exactly known in many contexts, since they are connected with anomalies, and thus can be used to compute string/quantum corrections to the AdS Chern-Simons terms. This is explained in [22, 23], where these results are used to give a simple derivation of higher derivative corrections to black hole entropy.

6. Example: $AdS_3 \times S^3$

In this section we derive an expression for the $SO(4)$ charges of asymptotically $AdS_3 \times S^3$ geometries. The analysis consists of a careful treatment of the 3-form field strength. We find it necessary to add a gauge dependent term to the naive expression for G_3 , and then show how this term gives rise to nonzero conserved charges.

The action is

$$S = \frac{1}{16\pi G_6} \int d^6x (\sqrt{g}R + \frac{1}{2} \star G_3 \wedge G_3) , \tag{6.1}$$

and the background $AdS_3 \times S^3$ solution is

$$\begin{aligned} ds^2 &= ds_{AdS_3}^2 + ds_{S^3}^2 \\ G_3 &= Q(\epsilon_{S^3} + i \star^6 \epsilon_{S^3}) \end{aligned} \tag{6.2}$$

where ϵ_{S^3} is the volume form on the unit 3-sphere. The factor of i comes from working in Euclidean signature.

6.1 Ansatz for asymptotic form of G_3

We now assume that the asymptotic metric takes the form (3.1), and seek an expression for the asymptotic form of G_3 . As in the previous section, we start by demanding that G_3 be closed, be gauge invariant, have a fixed integral over the S^3 fiber, and reduces to (6.2) when $A^{ij} = 0$. Our first guess is therefore

$$G_3 = Q(2\pi^2 e_3 + i \star^6 \epsilon_{S^3}) + dC_{AdS} , \quad (\text{first guess}) . \tag{6.3}$$

The volume form on S^3 is defined as $\epsilon_{S^3} = \frac{1}{3!} \epsilon_{ijkl} Dy^i Dy^j Dy^k y^l$. The contribution dC_{AdS} representing fluctuations of the AdS part will play no role in our discussion, and will be suppressed henceforth.

This expression indeed satisfies the conditions stated above. But it suffers from an important flaw. We require an expression not just for G_3 , but also for its potential C_2 . A glance at the explicit expression (4.3) for e_3 shows that it contains terms cubic in A^{ij} with no derivatives. This makes it clear that if we try to write $G_3 = dC_2$ we will be forced to write a nonlocal expression for C_2 . This nonlocality is troublesome when we recall that branes will couple directly to C_2 and hence be described by nonlocal actions. While we

might perhaps be able to make sense of this, it seems preferable to seek a modification of (6.3) compatible with a local expression for C_2 .

The root of the problem is that the closure of e_3 is in some sense an accident. In general, the global angular form e_3 is defined so that $de_3 = \chi_4$, the Euler class of the sphere bundle. If the dimension of the base AdS is less than 4, χ_4 trivially vanishes, since χ_4 is defined as a 4-form on the base space alone. In higher dimensional AdS spaces we are forced by the closure of G to write

$$G_3 = Q (2\pi^2(e_3 - \chi_3) + i \star^6 \epsilon_{S^3}) , \tag{6.4}$$

where we have taken advantage of the closure of χ_4 to write $\chi_4 = d\chi_3$. This expression is closed in any dimension and allows for the construction of a local ansatz for C_2 as we will see below. We will therefore take (6.4) as our asymptotic form for G_3 .

On the other hand, recall that the original motivation for writing (6.3) was based on the gauge invariance of G_3 , yet in (6.4) we have just added a term to our ansatz which is gauge dependent. This will not be a problem provided that the gauge variation of the action is a pure boundary term, since then the equations of motion will still be gauge invariant. We now show that this is indeed the case.

We write the gauge variation of χ_3 as $\delta_\Lambda \chi_3 = d\chi_2$, so that

$$\delta_\Lambda G_3 = -2\pi^2 Q d\chi_2 . \tag{6.5}$$

The variation of the action is then

$$\delta_\Lambda S = \frac{\pi Q}{8G_6} \int_{\partial} \star G_3 \wedge \chi_2 - \frac{\pi Q}{8G_6} \int d \star G_3 \wedge \chi_2 . \tag{6.6}$$

The second term vanishes⁷, leaving just the following boundary term:

$$\delta_\Lambda S = \frac{\pi Q}{8G_6} \int_{\partial} \star G_3 \wedge \chi_2 = \frac{i\pi^3 Q^2}{4G_6} \int_{\partial AdS} \chi_2 . \tag{6.7}$$

This is of course simply the gauge variation of a three dimensional Chern-Simons term

$$\begin{aligned} S_{CS} &= \frac{i\pi^3 Q^2}{4G_6} \int_{AdS} \chi_3 \\ &= -\frac{ik}{4\pi} \int_{AdS} \text{Tr}(A_L dA_L + \frac{2}{3} A_L^3) + \frac{ik}{4\pi} \int_{AdS} \text{Tr}(A_R dA_R + \frac{2}{3} A_R^3) , \end{aligned} \tag{6.8}$$

where we used (4.4), and defined

$$k = \frac{\pi^2 Q^2}{8G_6} . \tag{6.9}$$

⁷To see this, note that χ_2 has both legs along the AdS, and is constant on the sphere. Therefore, this term contains a factor of $\int_{S^3} d\star G_3$. Now decompose the exterior derivative as: $d = d_{S^3} + d_{AdS}$. Then, since $\star G_3$ is globally defined, we have $\int_{S^3} d_{S^3} \star G = 0$. Finally, we need $d_{AdS} \int_{S^3} \star G = 0$. But the part of G_3 with all 3 AdS legs is $\star \epsilon_{S^3}$ (using that χ_3 vanishes on the boundary for solutions obeying (2.3) and $A_\eta = 0$), so inside the integral we can take $\star G_3 = \epsilon_{S^3}$. The integral of the volume form has no AdS dependence, so it is annihilated by d_{AdS} .

Our formula (6.4) for G_3 may seem a bit surprising, but we have shown that its variation is consistent with that of a Chern-Simons term in the three dimensional action. By contrast, the naive version (6.3) has vanishing gauge variation. In the charge analysis that follows, it will be clear that only the modified version (6.4) will give the desired results. It is also worth noting that our approach generalizes quite easily to the case of $AdS_{2n-1} \times S^{2n-1}$. However, we should note that since we are working in a Lagrangian formalism we cannot immediately include examples with self-dual field strength, such as $AdS_5 \times S^5$ in IIB supergravity. To cover these cases we should instead work with the equations of motion; we hope to return to this in the future.

6.2 Comment on chiral anomalies

We can now give an illuminating higher dimensional interpretation of chiral anomalies in this context. In (6.7) we showed that the action is not invariant under gauge transformations that extend to the boundary; this is the anomaly. The explanation of this is that from the six-dimensional point of view it is clear that these are not gauge transformations, since they shift G_3 according to (6.5). We further observe that $\delta_\Lambda S = 0$ when χ_2 is an exact form, which is when $\delta_\Lambda G_3 = 0$. So the true (non-anomalous) gauge symmetries of the three-dimensional theory are just those that are manifest gauge symmetries of the six-dimensional theory.

6.3 Variation of the action

We now derive the current by computing the on-shell variation of the action with respect to A^{ij} . The variation of the Einstein-Hilbert term in (6.1) is proportional to the field strength of A^{ij} ; there is no contribution to the current since the potentials are flat at the boundary. Thus we need only consider

$$\delta S = -\frac{1}{16\pi G_6} \int_{\partial} \star G_3 \wedge \delta C_2 . \tag{6.10}$$

To proceed we need an expression for C_2 with G_3 given in (6.4). The basic formula needed for this is⁸

$$e_3 - \chi_3 = \frac{1}{3!} \epsilon_{ijkl} dy^i dy^j dy^k y^l + d \left[\frac{1}{4\pi^2} \epsilon_{ijkl} A^{ij} dy^k y^l - \frac{1}{8\pi^2} \epsilon_{ijkl} A^{ij} A^{km} y^l y^m \right] . \tag{6.11}$$

Now, using the explicit expression for e_3 given in (4.3), together with the fact that χ_3 vanishes at the boundary,⁹ the variation of the action has two terms

$$\begin{aligned} \delta S = & -\frac{iQ^2}{128\pi G_6} \int_{\partial} \epsilon_{mnpq} \epsilon_{ijkl} dy^m dy^n dy^k y^r y^q y^l A^{pr} \delta A^{ij} \\ & -\frac{iQ^2}{728\pi G_6} \delta \int_{\partial} \epsilon_{mnpq} dy^m dy^n dy^p y^q \epsilon_{ijkl} A^{ij} A^{km} y^l y^m . \end{aligned} \tag{6.12}$$

⁸To verify this formula it is helpful to use the SO(4) invariance to set, say, $y^4 = 1$.

⁹We are assuming the potentials are of the form (2.3).

The second line of (6.12) is easily seen to vanish: under the integral we can replace $y^l y^m \rightarrow \frac{1}{4} \delta^{lm}$, and then use $\epsilon_{ijkl} A^{ij} A^{kl} = 0$. The first line is straightforward, though a bit tedious, to work out. After performing the dy^i integration we find

$$\delta S = -\frac{ik}{16\pi} \int_{\partial AdS} \epsilon_{ijkl} A^{ij} \delta A^{kl}, \quad (6.13)$$

with k given in (6.9). Converting to $SU(2)_L \times SU(2)_R$ using the conventions in the appendix, the variation can be written

$$\delta S = -\frac{ik}{8\pi} \int_{\partial AdS} (A_L^a \delta A_L^a - A_R^a \delta A_R^a). \quad (6.14)$$

The remainder of the analysis now essentially reduces to that of section 2, taking into account the fact that we have two copies of $SU(2)$ gauge fields appearing with opposite sign. The boundary term analogous to (2.6) is therefore

$$S_{\text{bdy}}^{\text{gauge}} = -\frac{k}{16\pi} \int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta} (A_{L\alpha}^a A_{L\beta}^a + A_{R\alpha}^a A_{R\beta}^a). \quad (6.15)$$

The currents are

$$\begin{aligned} J_{Lw}^a &= \frac{ik}{2} A_{Lw}^a, & J_{L\bar{w}}^a &= 0 \\ J_{Rw}^a &= 0, & J_{R\bar{w}}^a &= \frac{ik}{2} A_{R\bar{w}}^a. \end{aligned} \quad (6.16)$$

The charges

$$J_{L0}^a = \oint \frac{dw}{2\pi i} J_{Lw}^a, \quad J_{R0}^a = -\oint \frac{d\bar{w}}{2\pi i} J_{R\bar{w}}^a, \quad (6.17)$$

then obey the $SU(2)_L \times SU(2)_R$ algebra.

6.4 Example of spectral flow

We start with global $AdS_3 \times S^3$

$$ds^2 = (1 + r^2/\ell^2) dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\xi^2 + \ell^2 (d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\phi^2). \quad (6.18)$$

The angular coordinates are related to the y^i as

$$\begin{aligned} y^1 &= \sin \theta \sin \psi \\ y^2 &= \sin \theta \cos \psi \\ y^3 &= \cos \theta \sin \phi \\ y^4 &= \cos \theta \cos \phi, \end{aligned} \quad (6.19)$$

and we also define the complex AdS_3 boundary coordinate

$$w = \xi + it/\ell. \quad (6.20)$$

We now consider diffeomorphisms that implement a spectral flow:

$$d\psi \rightarrow d\psi + \eta_L dw + \eta_R d\bar{w}, \quad d\phi \rightarrow d\phi + \eta_L dw - \eta_R d\bar{w}. \quad (6.21)$$

To preserve the periodicities we require $2\eta_{L,R} \in \mathbb{Z}$. This transformation induces the following gauge fields

$$A^{12} = -\eta_L dw + \eta_R d\bar{w}, \quad A^{34} = -\eta_L dw - \eta_R d\bar{w}, \quad (6.22)$$

or equivalently,

$$A_L^3 = 2\eta_L dw, \quad A_R^3 = 2\eta_R d\bar{w}. \quad (6.23)$$

The charges are therefore

$$J_{L0}^3 = k\eta_L, \quad J_{R0}^3 = k\eta_R. \quad (6.24)$$

These are the correct charges induced by spectral flow (as in (4.5).) Following the analysis of section 2, we also find that the Virasoro charges transform as in (4.5). This example therefore provides a simple illustration of how coordinate transformations can generate nonzero charges.

6.5 General rotating solutions of D1-D5 system

The D1-D5 system is the canonical example of an $\text{AdS}_3 \times S^3$ geometry. Comparing normalizations with, e.g. [24], we find

$$k = N_1 N_5. \quad (6.25)$$

This agrees with level of the $\text{SU}(2)$ current algebras of the dual CFT.

General solutions corresponding to black holes, black rings, or otherwise, take the asymptotic form (3.1). We typically choose coordinates such that the nonzero $\text{SU}(2)_L \times \text{SU}(2)_R$ charges are $J_{L,R0}^3 = \frac{1}{2}J_{L,R}$, with $J_{L,R} \in \mathbb{Z}$. These solutions therefore have

$$A_L^3 = \frac{J_L}{k} dw, \quad A_R^3 = \frac{J_R}{k} d\bar{w}. \quad (6.26)$$

These charges are conserved provided that (as is the case for the standard black hole/ring solutions) $A_{L\bar{w}}^a = A_{Rw}^a = 0$ on the boundary. If these components are nonzero then the currents are anomalous, and the charges are not conserved. This is completely consistent with the $\text{AdS}_3/\text{CFT}_2$ dictionary, in particular with the R-symmetry anomalies of the CFT.

6.6 Generalization to $\text{AdS}_{2n-1} \times S^{2n-1}$

The generalized version of (6.4) is

$$G_{2n-1} = Q (\Omega_{2n-1}(e_{2n-1} - \chi_{2n-1}) + i \star^{4n-2} \epsilon_{S^{2n-1}}), \quad (6.27)$$

where the Euler class is $\chi_{2n} = d\chi_{2n-1}$. Following the same steps as led to (6.8), we find that the $2n - 1$ dimensional action contains the Chern-Simons term

$$S_{CS} = \frac{i\Omega_{2n-1}^2 Q^2}{16\pi G_{4n-2}} \int_{\text{AdS}} \chi_{2n-1}. \quad (6.28)$$

A contribution to the current is obtained from the on-shell variation of (6.28). Two other contributions to the current come from the option of adding a boundary term analogous to (6.15), and from the variation of Maxwell type terms (note that above three dimensions, the Chern-Simons term is no longer the term with the fewest derivatives.) We will not explore this further here.

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A. Translation between $SO(4)$ and $SU(2)_L \times SU(2)_R$

Our $SO(4)$ generators are

$$J^{ij} = -i(y^i \partial_j - y^j \partial_i) . \quad (\text{A.1})$$

We then define self-dual and anti-self dual combinations:

$$\begin{aligned} J_+^{ij} &= \frac{1}{2} \left(\frac{1}{2} \epsilon^{ijkl} J^{kl} + J^{ij} \right) \\ J_-^{ij} &= \frac{1}{2} \left(\frac{1}{2} \epsilon^{ijkl} J^{kl} - J^{ij} \right) . \end{aligned} \quad (\text{A.2})$$

In terms of these we define the generators ($a = 1, 2, 3$)

$$\begin{aligned} J_L^a &= J_+^a \\ J_R^a &= J_-^a \end{aligned} \quad (\text{A.3})$$

which obey the $SU(2)_L \times SU(2)_R$ algebra:

$$\begin{aligned} [J_L^a, J_L^b] &= i \epsilon^{abc} J_L^c \\ [J_R^a, J_R^b] &= i \epsilon^{abc} J_R^c \\ [J_L^a, J_R^b] &= 0 . \end{aligned} \quad (\text{A.4})$$

The $SO(4)$ gauge fields A^{ij} are then related to the $SU(2)_L \times SU(2)_R$ gauge fields $A_{L,R}^a$ via

$$A_L^a J_L^a + A_R^a J_R^a = \frac{1}{2} A^{ij} J^{ij} \quad (\text{A.5})$$

which yields

$$A^{a4} = -\frac{1}{2} (A_L^a - A_R^a), \quad A^{ab} = -\frac{1}{2} \epsilon^{abc} (A_L^c + A_R^c) . \quad (\text{A.6})$$

Upon defining $A_{L,R} = A_{L,R}^a \sigma^a$, we find that (A.6) implies the relations

$$\begin{aligned} F^{ij} F^{ij} &= -2 \text{Tr} F_L^2 - 2 \text{Tr} F_R^2 \\ \chi_4 &= \frac{1}{32\pi^2} \epsilon^{ijkl} F^{ij} F^{kl} = -\frac{1}{8\pi^2} \text{Tr} F_L^2 + \frac{1}{8\pi^2} \text{Tr} F_R^2 \\ \chi_3 &= -\frac{1}{8\pi^2} \text{Tr} (A_L dA_L + \frac{2}{3} A_L^3) + \frac{1}{8\pi^2} \text{Tr} \left(A_R dA_R + \frac{2}{3} A_R^3 \right) \end{aligned} \quad (\text{A.7})$$

with $\chi_4 = d\chi_3$.

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